An extension of the linearized theory of supersonic flow past quasi-cylindrical bodies, with applications to wing–body interference

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(*Received* 24 *October* 1957)

SUMMARY

An extension of the linearized theory of supersonic flow past quasi-cylindrical bodies of almost circular cross-section has been found which enables a direct calculation to be made of the overall forces on wings mounted on such bodies, subject to certain restrictions on the plan-form. The method is applied to two examples: (i) the effect of an arbitrary body distortion on static stability at supersonic speeds; and (ii) the effect of wing–body interference on rectangular wings mounted on a cylindrical body. The drag calculations in the second example are compared with the results of the supersonic area rule, which is found to be in error for moderate values of the ratio of wing chord to body radius, though the discrepancy is not serious from a practical point of view.

1. INTRODUCTION

The linearized theory of supersonic flow past quasi-cylindrical bodies of circular cross-section, initially due to Lighthill and Ward (see, for example, Ward 1955), has recently been developed by Randall (1955) to cover the case when the cross-section of the body is of arbitrary shape differing only slightly from a circle. A similar method has also been used by Nielsen (1955) in his work on wing–body interference. The method consists briefly of expanding the local streamwise slope of the body, together with a suitable basic solution of the linearized equation for the velocity potential, as Fourier series in terms of the meridian angle; the boundary condition of zero normal velocity, which may be considered to apply at a mean circular cylinder, is then used to determine the arbitrary coefficients in the series for the velocity potential. The Heaviside operational calculus is found to be invaluable in simplifying the analysis. The complete pressure field due to a body of this type can thus be found, at least in principle, and indeed the calculation of the pressure on the surface of the body is now relatively simple. But, as Randall (1955) points out, a very much larger amount of numerical work is necessary if the pressure off the body surface is required, and although much of the basic computation has now been done by Mersman (1954) and Nielsen (1957), there is still quite a formidable task remaining in any particular case.

Such extensive pressure calculations are necessary if it is desired to find the effect of the flow round quasi-cylindrical bodies on wings or fins; and in effect they are also necessary in problems involving the interference potential between wings and bodies. If *detailed* wing pressure distributions are needed, there seems little that can be done by way of further simplification. However, if only the overall forces or moments on the wings are required, it has been found that in certain special cases considerable analytical simplification can be achieved and the problem reduced to one which is little more difficult or lengthy than that of determining the pressure on the body surface. This has been done by integrating the disturbance pressure field due to each Fourier component of the body distortion along lines in the plane of the wings at right angles to the axis of the body, from the body surface to the limit of the disturbance. The restrictions on the wing shape which must apply in order that the method shall be valid therefore differ according to the particular force or moment in question. In all cases the leading edge must be supersonic and the trailing edge at right angles to the free stream; if there are subsonic wing tips these must be outside the field of influence of the body distortion. In the case of drag, it is further necessary that the local wing slope should be constant along lines at right angles to the body axis, and this effectively limits the method to rectangular wings of constant cross-section.

In spite of these rather severe limitations, the method has direct applications to some problems which are of practical interest. First, it is shown how to obtain the changes in lift and pitching or rolling moment produced by an arbitrary small distortion of body shape. Secondly, the more conventional problem of wing–body interference is considered, and results are obtained for the lift, pitching moment and drag of rectangular wings of arbitrary cross-section mounted at incidence on an infinite circular cylinder parallel to the free stream direction. They are compared with the previous work of Nielsen & Pitts (1952) and Nielsen (1955), which is shown to contain errors for the larger values of the ratio of the wing chord to the body radius.

The results obtained in this way for the wave-drag of combinations with rectangular wings and cylindrical bodies are of particular interest because they provide an example in which direct comparison may be made with the supersonic area rule (Jones 1953). It appears that for moderate values of the chord–radius ratio the interference correction to the total wave-drag given by the area rule is actually of the wrong sign, and that only for very large values of this ratio does it approach the true linearized theory. It must, however, be realized that this interference correction is quite small compared with the total wing drag (except possibly for wings of very small aspect ratio, to which the present method does not apply); it is probable that in most problems involving real aeroplane shapes the area rule will be much more successful in estimating the total wave-drag, but this point still requires further investigation.

2. Linearized theory of supersonic flow past quasi-cylindrical bodies

We consider supersonic flow with a free stream velocity U past a body whose shape differs only slightly from an infinite circular cylinder of radius R_0, with its axis in the direction of the free stream. We shall use standard Cartesian axes Ox, Oy and Oz, with Ox coincident with the axis of the cylinder; Oy will later be taken in the plane of the wings and Oz vertically upward. The origin O will normally be chosen so that the body distortion is zero upstream from the plane Oyz (see figure 1). Dimensional coordinates will be denoted by (X, Y, Z); non-dimensional coordinates (x, y, z) are defined by

$$x = X/\beta R_0, \qquad y = Y/R_0, \qquad z = Z/R_0,$$

where $\beta = (M^2-1)^{1/2}$ and M is the Mach number of the free stream. We shall also use polar coordinates (R, θ, X) such that

$$Y = R\sin\theta, \qquad Z = R\cos\theta,$$

and again define $r = R/R_0$.

Under the usual limitations of the linearized theory, the motion has a velocity potential $UX+\phi(X, Y, Z)$, where the disturbance potential ϕ satisfies the equation

$$\beta^2\frac{\partial^2\phi}{\partial X^2} = \frac{\partial^2\phi}{\partial Y^2}+\frac{\partial^2\phi}{\partial Z^2} = \frac{\partial^2\phi}{\partial R^2}+\frac{1}{R}\frac{\partial\phi}{\partial R}+\frac{1}{R^2}\frac{\partial^2\phi}{\partial\theta^2}. \tag{1}$$

Introduction of the dimensionless variables defined above reduces this to

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}. \tag{2}$$

We shall use the Heaviside operational calculus (see, for example, Jeffreys & Jeffreys (1956) or van der Pol & Bremmer (1950)) in which p stands for $\partial/\partial x$, so that equation (2) may be written

$$p^2\boldsymbol{\varphi} = \frac{\partial^2\boldsymbol{\varphi}}{\partial r^2}+\frac{1}{r}\frac{\partial\boldsymbol{\varphi}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\boldsymbol{\varphi}}{\partial\theta^2} \qquad (3)^*$$

(see Ward (1955) for a rigorous justification of this procedure).

The basic solution of equation (3) that is applicable in the present case is

$$\boldsymbol{\varphi}(p, r, \theta) = \sum_{n=0}^{\infty} K_n(pr)\{a_n(p)\cos n\theta+b_n(p)\sin n\theta\}, \tag{4}$$

where K_n is the Bessel function of the second kind with purely imaginary argument (as defined in Watson (1944), p. 78), and $a_n(p)$, $b_n(p)$ are arbitrary functions of p.

* In what follows the operational form of a function $f(x)$ will be denoted by $\mathbf{f}(p)$, where

$$\mathbf{f}(p) = p\int_0^{\infty} e^{-px}f(x)\,dx,$$

and the relationship between $f(x)$ and $\mathbf{f}(p)$ will also be written (cf. van der Pol & Bremmer 1950)

$$\mathbf{f}(p) \subset f(x) \quad \text{or} \quad f(x) \supset \mathbf{f}(p).$$

The boundary condition of zero normal velocity at the surface of the body gives

$$\left.\frac{\partial\phi}{\partial R}\right|_{R=R_0} = U\eta(X,\theta), \tag{5}$$

where η is the slope of a meridian section of the body and is assumed to be zero for $X < 0$. The non-dimensional form of equation (5) is

$$\frac{1}{UR_0}\left.\frac{\partial\phi}{\partial r}\right|_{r=1} = \eta(x,\theta). \tag{6}$$

We may expand $\eta(x,\theta)$ as a Fourier series in θ:

$$\eta(x,\theta) = \sum_{n=0}^{\infty}\{A_n(x)\cos n\theta + B_n(x)\sin n\theta\}H(x) \tag{7}$$

over the range $0 \leqslant \theta \leqslant 2\pi$, where $H(x)$ is the Heaviside unit function. The operational form of this is

$$\boldsymbol{\eta}(p,\theta) = \sum_{n=0}^{\infty}\{\mathbf{A}_n(p)\cos n\theta + \mathbf{B}_n(p)\sin n\theta\}. \tag{8}$$

Combining equations (4), (6) and (8) and equating coefficients of $\cos n\theta$ and $\sin n\theta$, we find that

$$a_n(p) = UR_0\,\mathbf{A}_n(p)/pK_n'(p),$$

and

$$b_n(p) = UR_0\,\mathbf{B}_n(p)/pK_n'(p).$$

Substituting these values in equation (4), we obtain for the final operational form of the disturbance potential

$$\boldsymbol{\varphi}(p,r,\theta) = UR_0\sum_{n=0}^{\infty}\frac{K_n(pr)}{pK_n'(p)}\{\mathbf{A}_n(p)\cos n\theta + \mathbf{B}_n(p)\sin n\theta\}. \tag{9}$$

The function $\phi(x,r,\theta)$ which is the interpretation of equation (9) satisfies the boundary conditions of the body and at infinity, and is identically zero for $x < r$; it is therefore the correct linearized disturbance potential over the whole of space exterior to the body.

The perturbation pressure coefficient $(P-P_0)/q_0$ (where $q_0 = \frac{1}{2}\rho_0 U^2$, ρ_0 denoting the density of the free stream) is given to the same order of approximation by the linearized expression

$$\begin{aligned} C_p &= -\frac{2}{U}\frac{\partial\phi}{\partial X} \supset -\frac{2}{\beta R_0 U}p\boldsymbol{\varphi} \\ &= -\frac{2}{\beta}\sum_{n=0}^{\infty}\frac{K_n(pr)}{K_n'(p)}\{\mathbf{A}_n(p)\cos n\theta + \mathbf{B}_n(p)\sin n\theta\}. \end{aligned} \tag{10}$$

If the functions $W_n(x,r)$ are defined by

$$W_n(x,r) \supset \mathbf{W}_n(p,r) = -e^{p(r-1)}K_n(pr)/K_n'(p), \tag{11}$$

and we interpret equation (10) by the product theorem (Jeffreys & Jeffreys 1956, § 12.13), we obtain

$$\begin{aligned} C_p = \frac{2}{\beta}\sum_{n=0}^{\infty}\Big\{\cos n\theta\int_{\xi=r-1-}^{x} W_n(x-\xi,r)\,dA_n(\xi-r+1) + \\ +\sin n\theta\int_{\xi=r-1-}^{x} W_n(x-\xi,r)\,dB_n(\xi-r+1)\Big\}, \end{aligned} \tag{12}$$

where the Stieltjes notation has been used. Again, equation (12) gives the perturbation pressure everywhere exterior to the body.

Randall (1955) has evaluated the functions $W_n(x) \equiv W_n(x, 1)$ for $r = 1$ and $n = 0-10$ over a wide range of values of x; the determination of the pressure on the surface of the body is thus a comparatively simple matter. In cases where $W_n(x, r)$ are required for $r > 1$, the computational problem is much more formidable, though considerable progress has been made by Mersman (1954) and Nielsen (1955, 1957). The functions $\mathbf{W}_n(p, r)$ defined above (which differ only by the factor $e^{p(r-1)}$ from those used by Randall (1955)) are related to the corresponding functions of Nielsen (1955) (which will be distinguished here by the additional suffix N) by

$$\int_0^x W_{n,N}(\xi, r)\, d\xi = r^{-1/2} - W_n(x, r) \quad \text{for } x \geqslant 0, \tag{13}$$

and in particular, when $r = 1$,

$$\int_0^x W_{n,N}(\xi)\, d\xi = 1 - W_n(x). \tag{14}$$

The results of Mersman and of Nielsen can be used, in conjunction with equations (7) and (12), to obtain the complete pressure field of any quasi-cylindrical body.

Unfortunately, it is found that in most cases of practical importance (see, for example, Nielson 1955) the Fourier series (12) does not converge with sufficient rapidity for small values of $(x-r+1)$, so that the pressure near the Mach cone $x = r-1$ is difficult to determine accurately. This fact, together with the large amount of numerical work which still has to be done in any given practical case, means that the method is not to be recommended as a way of determining overall wing forces if there is any reasonable alternative.

In the next section we shall show that it is possible to obtain comparatively simple expressions, which do not in fact require any knowledge of the functions $W_n(x, r)$ except at $r = 1$, for the integrals of the pressure coefficient (as given by equation (10) or (12)) from $r = 1$ to ∞ along strips perpendicular to the x-axis. In the following sections some applications to practical problems will be given.

3. Determination of certain pressure integrals

3.1. *Definition of the functions $I_{m,n}$*

In problems requiring a knowledge of wing lift and rolling moment, it is necessary to determine both the total normal force and the moment about Ox produced by the disturbance pressure field acting on an infinitesimal strip, parallel to Oy in the plane $z = 0$, extending from the surface of the cylinder $R = R_0$ to the Mach lines $R = R_0 + X/\beta$ which represent the limit of the disturbance, and of width δX (see figure 1). These are given respectively by

$$\delta L = -\delta X \int_{R_0}^{R_0+X/\beta} (P-P_0)\,dR = -R_0\,\delta X \int_1^{1+x} (P-P_0)\,dr,$$

and

$$\delta M_x = -\delta X \int_{R_0}^{R_0+X/\beta} R(P-P_0)\,dR = -R_0^2\,\delta X \int_1^{1+x} r(P-P_0)\,dr.$$

Since $P-P_0 = 0$ for $r > 1+x$, the upper limit in these expressions may be extended to infinity.

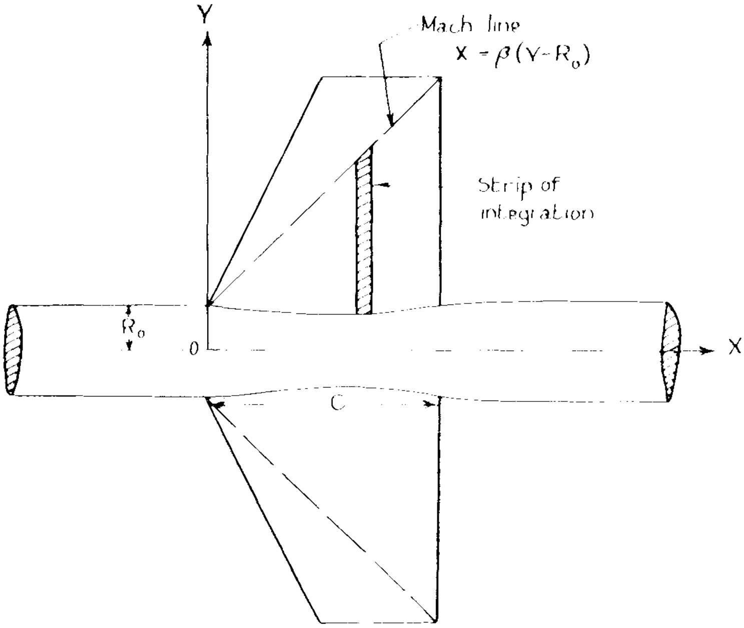


Figure 1. Typical wing-body combination.

Using equation (10) for $(P-P_0)$, we obtain

$$R_0^{-1} q_0^{-1}\,\partial L/\partial X \supset (2/\beta) \sum_{n=0}^{\infty} \mathbf{A}_n(p) \int_{r=1}^{\infty} K_n(pr)\,dr / K_n'(p), \tag{15}$$

and

$$R_0^{-2} q_0^{-1}\,\partial M_x/\partial X \supset (2/\beta) \sum_{n=0}^{\infty} \mathbf{A}_n(p) \int_{r=1}^{\infty} r K_n(pr)\,dr / K_n'(p), \tag{16}$$

where $q_0 = \frac{1}{2}\rho_0 U^2$.

These may be written in the form

$$R_0^{-1} q_0^{-1}\,\partial L/\partial X \supset -(2/\beta) \sum_{n=0}^{\infty} p\mathbf{A}_n(p)\mathbf{I}_{0,n}(p) \tag{17}$$

and

$$R_0^{-2} q_0^{-1}\,\partial M_x/\partial X \supset -(2/\beta) \sum_{n=0}^{\infty} p\mathbf{A}_n(p)\mathbf{I}_{1,n}(p), \tag{18}$$

where

$$\mathbf{I}_{m,n}(p) = -\int_{r=1}^{\infty} r^m K_n(pr)\,dr / \{pK_n'(p)\}. \tag{19}$$

The remainder of this section will be devoted to the interpretation of the operational functions $\mathbf{I}_{m,n}(p)$.

3.2. *Interpretation of* $\mathbf{I}_{m,n}(p)$

The transformation $pr = \xi$ in equation (19) gives

$$\mathbf{I}_{m,n}(p) = -\int_p^\infty \xi^m K_n(\xi)\, d\xi / \{p^{m+2} K_n'(p)\}. \tag{20}$$

The upper limit in equation (20) should strictly be $p\infty$; but since when interpreting $\mathbf{I}_{m,n}(p)$ by means of Bromwich's integral (Jeffreys & Jeffreys 1956, § 12.09) the real part of p is to be taken as positive, and since also $(K_n\xi) \sim (\pi/2\xi)^{1/2} e^{-\xi}$ when $|\xi| \to \infty$, the integrals converge at the upper limit and are given correctly as written.

The functions $\int_p^\infty \xi^m K_n(\xi)\, d\xi$ may be expressed in terms of the associated Bessel functions known as Lommel functions (see Watson 1944, § 10.7 *et seq.*). We have (Watson 1944, § 10.74)

$$\int^z z^m C_m(z)\, dz = (m+n-1) z C_n(z) S_{m-1,n-1}(z) - z C_{n-1}(z) S_{m,n}(z), \tag{21}$$

where $C_n(z)$ is any cylinder function (arbitrary solution of Bessel's equation) and $S_{m,n}(z)$ is the Lommel function of the second kind. In particular, taking $C_n(z)$ to be the first Hankel function $H_n^{(1)}(z)$, and writing $z = i\xi$, we get

$$i^{m+1} \int^\xi \xi^m H_n^{(1)}(i\xi)\, d\xi = (m+n-1) i\xi H_n^{(1)}(i\xi) S_{m-1,n-1}(i\xi) - i\xi H_{n-1}^{(1)}(i\xi) S_{m,n}(i\xi). \tag{22}$$

Now $K_n(\xi) = \frac{1}{2}\pi i^{n+1} H_n^{(1)}(i\xi)$, (Watson 1944, § 3.7), so that

$$\int^\xi \xi^m K_n(\xi)\, d\xi = -[(m+n-1)\xi K_n(\xi) \mathbf{T}_{m-1,n-1}(\xi) + \xi K_{n-1}(\xi) \mathbf{T}_{m,n}(\xi)], \tag{23}$$

where

$$\mathbf{T}_{m,n}(\xi) = i^{-m+1} S_{m,n}(i\xi). \tag{24}$$

The Lommel function $S_{m,n}(z)$ is a particular integral of the differential equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - n^2) y = z^{m+1}. \tag{25}$$

When $m+n$ is odd, $S_{m,n}(z)$ is defined by the finite series

$$S_{m,n}(z) = z^{m-1}\left[1 - \frac{(m-1)^2 - n^2}{z^2} + \frac{\{(m-1)^2 - n^2\}\{(m-3)^2 - n^2\}}{z^4} - \dots\right]. \tag{26}$$

When $m+n$ is even, the definition of $S_{m,n}(z)$ is more complicated, but it still has the *asymptotic* expansion (26) when $|z|$ is large.

Thus $\mathbf{T}_{m,n}(z)$ is a particular integral of the differential equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} - (z^2 + n^2) y = -z^{m+1}, \tag{27}$$

and is given by

$$\mathbf{T}_{m,n}(z) = z^{m-1}\left[1 + \frac{(m-1)^2 - n^2}{z^2} + \frac{\{(m-1)^2 - n^2\}\{(m-3)^2 - n^2\}}{z^4} + \dots\right] \tag{28}$$

when $m+n$ is odd, while if $m+n$ is even it has the asymptotic expansion (28) when $|z|$ is large.

If in equation (23) we make use of the relation

$$-K_{n-1}(\xi) = K_n'(\xi)+n\xi^{-1}K_n(\xi),$$

we get

$$\int^{\xi} \xi^m K_n(\xi)\,d\xi = \xi K_n(\xi)\{n\xi^{-1}\mathbf{T}_{m,n}(\xi)-(m+n-1)\mathbf{T}_{m-1,n-1}(\xi)\}+ \\ +\xi K_n'(\xi)\mathbf{T}_{m,n}(\xi), \quad (29)$$

and in particular

$$-\int_p^{\infty} \xi^m K_n(\xi)\,d\xi = pK_n(p)\{np^{-1}\mathbf{T}_{m,n}(p)-(m+n-1)\mathbf{T}_{m-1,n-1}(p)\}+ \\ +pK_n'(p)\mathbf{T}_{m,n}(p),$$

since the indefinite integral given by (29) clearly vanishes when $\xi \to \infty$. Substituting in equation (20), and remembering that $\mathbf{W}_n(p) = -K_n(p)/K_n'(p)$ (equation (11) with $r = 1$), we obtain

$$\mathbf{I}_{m,n}(p) = p^{-m-1}\mathbf{T}_{m,n}(p)-p^{-1}\mathbf{W}_n(p)\times \\ \times[np^{-m-1}\mathbf{T}_{m,n}(p)-(m+n-1)p^{-m}\mathbf{T}_{m-1,n-1}(p)]. \quad (30)$$

It is convenient to write

$$p^{-m-1}\mathbf{T}_{m,n}(p) = \boldsymbol{\tau}_{m,n}(p), \quad (31)$$

so that

$$\mathbf{I}_{m,n}(p) = \boldsymbol{\tau}_{m,n}(p)-p^{-1}W_n(p)\{n\boldsymbol{\tau}_{m,n}(p)-(m+n-1)\boldsymbol{\tau}_{m-1,n-1}(p)\}. \quad (32)$$

In order to interpret the operational functions $\mathbf{T}_{m,n}(p)$ and $\boldsymbol{\tau}_{m,n}(p)$ we first note that a series expression can be obtained at once from equation (28). Thus

$$\tau_{m,n}(x) \supset \boldsymbol{\tau}_{m,n}(p) = \frac{1!}{p^2}\left[1+\frac{(m-1)^2-n^2}{p^2}+ \right. \\ \left. +\frac{\{(m-1)^2-n^2\}\{(m-3)^2-n^2\}}{p^4}+\dots\right],$$

so that

$$\tau_{m,n}(x) = x^2/2!+\{(m-1)^2-n^2\}x^4/4!+\{(m-1)^2-n^2\}\{(m-3)^2-n^2\}x^6/6!+\dots \quad (33)$$

If $m+n$ is odd, this series terminates and completely defines $\tau_{m,n}(x)$. If $m+n$ is even, the series (33) still defines $\tau_{m,n}(x)$ for sufficiently small x (in fact for $x \leqslant 1$); but an alternative approach is preferable, which yields remarkably simple results in the cases of interest at present, namely $m = -1$, 0 and 1.

Consider the differential equation

$$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-n^2y = -\frac{x^k}{k!}, \quad (34)$$

where if k is a negative integer the right-hand side is to be replaced by zero. We take y to be that solution of equation (34) for $x > 0$ which satisfies the initial conditions $y(0) = A$, $y'(0) = B$. Then the operational form of this equation may be shown, with the aid of the relation (cf. van der Pol & Bremmer 1950, Chs. IV & X)

$$x^n y(x) \supset p(-d/dp)^n \mathbf{y}(p)/p, \tag{35}$$

to be

$$\left.\begin{aligned} p^2\frac{d^2\mathbf{y}}{dp^2}+p\frac{d\mathbf{y}}{dp}-(p^2+n^2)\mathbf{y} &= -p^{-k}-pB-p^2A \quad \text{if } k \geqslant 0, \\ &= -pB-p^2A \quad \text{if } k < 0. \end{aligned}\right\} \tag{36}$$

Comparing equation (36) with the equation (27) satisfied by $\mathbf{T}_{m,n}(p)$, we see that they are identical provided that $k = -(m+1)$ and that (*a*) if $m = -1$, we take $A = B = 0$; (*b*) if $m = 0$, we take $A = 0$, $B = 1$; (*c*) if $m = 1$, we take $A = 1$, $B = 0$. Now the further transformation $x = \sin\phi$ reduces equation (34) to the form

$$\frac{d^2y}{d\phi^2}+n^2y = \frac{\sin^k\phi}{k!}, \tag{37}$$

and the solution of equation (37) satisfying the required boundary conditions at $x = \phi = 0$ can easily be found.

In this way we obtain the results

$$\left.\begin{aligned} T_{-1,n}(x) &= n^{-2}(1-\cos n\phi), \\ T_{0\,n}(x) &= n^{-1}\sin n\phi, \\ T_{1,n}(x) &= \cos n\phi, \end{aligned}\right\} \quad 0 \leqslant x \leqslant 1 \tag{38}$$

where for $n = 0$ the limiting values as $n \to 0$ are to be taken.

The corresponding results for the functions $\tau_{m,n}(x)$ (equation (31)) are

$$\left.\begin{aligned} &\tau_{-1,n}(x) = n^{-2}(1-\cos n\phi), \quad (n \neq 0) \\ \text{with} \quad &\tau_{-1,0}(x) = \tfrac{1}{2}\phi^2; \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} &\tau_{0,n}(x) = \frac{1}{n^2-1}-\frac{1}{2n(n+1)}\cos(n+1)\phi-\frac{1}{2n(n-1)}\cos(n-1)\phi \quad (|n| > 1) \\ &\text{with} \quad \tau_{0,0}(x) = \cos\phi+\sin\phi-1 = (1-x^2)^{1/2}+x\sin^{-1}x-1, \\ &\text{and} \quad \tau_{0,1}(x) = \tfrac{1}{2}x^2; \end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} &\text{and } \tau_{1,n}(x) = \frac{1}{n^2-4}-\frac{1}{4(n-1)(n-2)}\cos(n-2)\phi-\frac{1}{2(n^2-1)}\cos n\phi- \\ &\qquad -\frac{1}{4(n+1)(n+2)}\cos(n+2)\phi, \quad (n > 2) \\ &\text{with} \quad \tau_{1,0}(x) = \tfrac{1}{2}x^2, \\ &\qquad \tau_{1,1}(x) = -\tfrac{1}{3}+\tfrac{1}{2}\pi\sin\phi+\tfrac{3}{8}\cos\phi-\tfrac{1}{24}\cos 3\phi, \\ &\text{and} \quad \tau_{1,2}(x) = \tfrac{1}{2}x^2-\tfrac{1}{8}x^4. \end{aligned}\right\} \tag{41}$$

It may be verified directly that equations (39) to (41) are equivalent to the series expressions (33) for $\tau_{m,n}(x)$.

It remains to consider what happens to $\tau_{m,n}(x)$ for $x > 1$ in cases where $\tau_{m,n}(x)$ is not simply a polynomial (i.e. when $m+n$ is even). In these cases the expressions (39) to (41) appear to break down, but it can be shown that they can still be used provided the real part is taken, so that ϕ is to be replaced by $\frac{1}{2}\pi$. Thus when $x > 1$, we have simply

$$\left.\begin{aligned} \tau_{-1,n}(x) &= n^{-2}, \qquad (n \text{ odd}) \\ \text{and} \qquad \tau_{-1,0}(x) &= \pi^2/8; \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned} \tau_{0,n}(x) &= (n^2-1)^{-1}, \quad (n \text{ even} \neq 0) \\ \text{and} \qquad \tau_{0,0}(x) &= \tfrac{1}{2}\pi x - 1; \end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned} \tau_{1,n}(x) &= (n^2-4)^{-1}, \quad (n \text{ odd} > 1) \\ \text{and} \qquad \tau_{1,1}(x) &= \tfrac{1}{4}\pi x - \tfrac{1}{3}. \end{aligned}\right\} \tag{44}$$

The range of values of m in equations (39) to (44) can be extended if desired by means of the recurrence relation

$$p^2\boldsymbol{\tau}_{m+2,n}(p) = 1 + \{(m+1)^2 - n^2\}\boldsymbol{\tau}_{m,n}(p), \tag{45}$$

whence

$$\tau_{m+2,n}(x) = \tfrac{1}{2}x^2 + \{(m+1)^2 - n^2\}\int_0^x (x-\xi)\tau_{m,n}(\xi)\,d\xi. \tag{46}$$

This may be derived either directly from equation (33) or from the recurrence relations satisfied by the Lommel functions.

Returning to equation (32) and interpreting, we get

$$I_{m,n}(x) = \tau_{m,n}(x) - \int_0^x W_n(x-\xi)\{n\tau_{m,n}(\xi) - (m+n-1)\tau_{m-1,n-1}(\xi)\}\,d\xi. \tag{47}$$

Equation (47), together with the expressions (33), (39) to (44) and (46) for $\tau_{m,n}(x)$, enables $I_{m,n}(x)$ to be calculated without reference to the values of $W_n(x, r)$ except for $r = 1$. This has been done for $m = 0$ and 1, $n = 0$ to 6 for a range of values of x from 0 to 5; the results are given in table 1, and are shown graphically in figure 2.

In spite of the comparative simplicity of equation (47), it does involve certain computational difficulties which are not immediately obvious. In the first place, if $m+n$ is *odd*, $\tau_{m,n}(x)$ is a polynomial of degree $m+n+1$, so that except for the smaller values of $m+n$, $I_{m,n}(x)$ is given by equation (47) as the difference of two functions which become large rapidly as x increases, while $I_{m,n}$ itself remains small. This fact, coupled with the oscillatory nature of $W_n(x)$ for the larger values of n, makes it difficult to obtain satisfactory accuracy in such cases; fortunately, asymptotic formulae are available which are adequate for most practical purposes (see below).

The second difficulty, which is less serious, arises from the fact that when $m+n$ is *even*, $\tau_{m,n}(x)$ has a discontinuity in its first or second derivative at $x = 1$; in fact $\tau'_{-1,n}(x)$ becomes infinite at this point. Thus care has to be taken in evaluating the integral in equation (47), and formulae of the type

x	$I_{0,0}$	$I_{0,1}$	$I_{0,2}$	$I_{0,3}$	$I_{0,4}$	$I_{0,5}$	$I_{0,6}$
0	0	0	0	0	0	0	0
0·2	0·0188		0·0185		0·0177		0·0166
0·4	0·0707		0·0675		0·0585		0·0457
0·5		0·1055		0·091		0·068	
0·6	0·1504		0·1361		0·0999		0·0574
0·8	0·2537		0·2143		0·1257		0·0469
1·0	0·3772	0·3545	0·2933	0·212	0·1307	0·068	0·0295
1·2	0·5181		0·3664		0·1188		0·0219
1·4	0·6741		0·4293		0·0996		0·0261
1·5		0·6686		0·234		0·037	
1·6	0·8432		0·4792		0·0822		0·0342
1·8	1·0240		0·5157		0·0721		0·0382
2·0	1·2150	0·9958	0·5396	0·194	0·0703	0·051	0·0363
2·2	1·4151		0·5523		0·0743		0·0328
2·4	1·6233		0·5660		0·0805		0·0312
2·5		1·3053		0·162			
2·6	1·8387		0·5530		0·0856		0·0321
2·8	2·0606		0·5456		0·0881		0·0336
3·0	2·2883	1·5814	0·5358	0·161	0·0879		0·0344
3·2	2·5213		0·5252		0·0860		0·0339
3·4	2·7590		0·5151		0·0838		0·0332
3·5		1·8185		0·170			
3·6	3·0010		0·5064		0·0821		0·0329
3·8	3·2470		0·4994		0·0815		0·0330
4·0	3·4966	2·0174	0·4944		0·0818		0·0333
4·2	3·7494		0·4912				
4·4	4·0053		0·4897				
4·6	4·2640		0·4895				
4·8	4·5252		0·4904				
5·0	4·7887		0·4917				
∞			0·5000	0·167	0·0833	0·050	0·0333

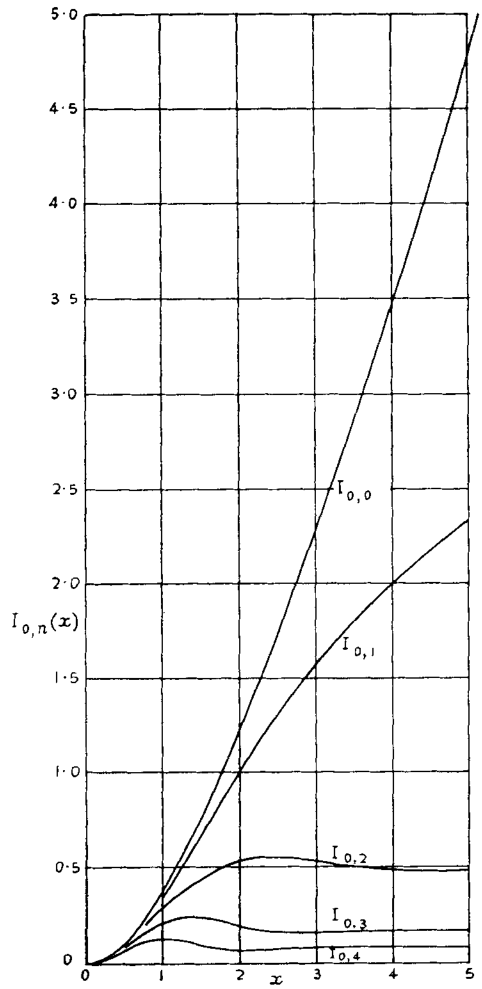
(*a*)

x	$I_{1,0}$	$I_{1,1}$	$I_{1,2}$	$I_{1,3}$	$I_{1,4}$	$I_{1,5}$	$I_{1,6}$
0	0	0	0	0	0	0	0
0·5	0·125	0·1229	0·1166	0·1067	0·094	0·080	0·065
1·0	0·5	0·4709	0·3926	0·2868	0·181	0·096	0·043
1·5	1·125	1·0002	0·1970	0·3718	0·150	0·056	0·037
2·0	2·0	1·6620	0·9411	0·3596	0·119	0·067	0·05
2·5	3·125	2·4129	1·1009	0·3303	0·127	0·070	
3·0	4·5	3·2184	1·1932	0·3201	0·130	0·064	
3·5	6·125	4·0527	1·2504	0·3273		0·067	
4·0	8·0	4·8984	1·2963	0·3340		0·067	
∞				0·3333	0·125	0·067	0·0417

(*b*)

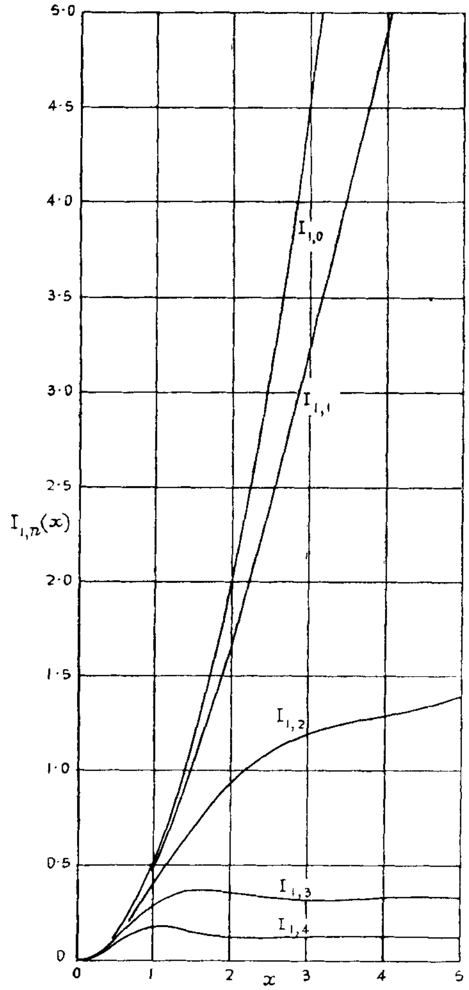
Table 1. The functions $I_{m,n}(x)$; (*a*) $I_{0,n}(x)$, (*b*) $I_{1,n}(x)$.

given in Jeffreys & Jeffreys (1956, § 9.092) must be used near $x = 1$ in place of conventional methods of integration. In spite of these singularities in the derivatives of $\tau_{m,n}(x)$, it can be shown that in fact $I_{m,n}(x)$ and all its derivatives are continuous functions of x for $x \geqslant 0$.



(a)

Figure 2 (a). The functions $I_{0,n}(x)$.



(b)

Figure 2 (b). The functions $I_{1,n}(x)$.

3.3. *Asymptotic expansions for $I_{m,n}(x)$*

When x is large, asymptotic expressions for the functions $I_{m,n}(x)$ can be obtained from the definition

$$I_{m,n}(x) \supset \mathbf{I}_{m,n}(p) = -p^{-n-2} \int_p^\infty \xi^m K_n(\xi)\, d\xi / K_n'(p)$$

by expanding as a series in p.

Now when $n > 0$,

$$K_n(\xi) = \tfrac{1}{2}[(n-1)!(\tfrac{1}{2}\xi)^{-n} - (2!)^{-1}(n-2)!(\tfrac{1}{2}\xi)^{-n+2} + \ldots + (-1)^{n-1}(\tfrac{1}{2}\xi)^{n-2}/(n-1)!] + (-1)^{n+1}(n!)^{-1}(\tfrac{1}{2}\xi)^n \times \{\log\tfrac{1}{2}\xi + \gamma - \tfrac{1}{2}(1+\tfrac{1}{2}+\ldots+n^{-1})\} + O(\xi^{n+1}\log\xi), \quad (48)$$

where γ is Euler's constant. We suppose first that $n > m+1$. It is convenient to denote that part of the series expansion of $\xi^m K_n(\xi)$ which involves only ***negative*** powers of ξ by $P(\xi^m K_n)$. Then $\xi^m K_n(\xi) - P\{\xi^m K_n(\xi)\}$ is regular at $\xi = 0$, and we can write

$$\int_p^\infty \xi^m K_n(\xi) = \int_0^\infty [\xi^m K_n(\xi) - P\{\xi^m K_n(\xi)\}]\,d\xi + \int_p^\infty P\{\xi^m K_n(\xi)\}\,d\xi - \int_0^p [\xi^m K_n(\xi) - P\{\xi^m K_n(\xi)\}]\,d\xi.$$

The first term of this series is a constant; and it is easy to see that the sum of the last two terms may be written simply as the indefinite integral $-\int^p \xi^m K_n(\xi)\,d\xi$, obtained by integrating the series for $\xi^m K_n(\xi)$ formally term by term.

The result differs slightly accordingly as $m+n$ is even or odd. In the former case the important terms of the expansion in ascending powers of p are

$$-\int_p^\infty \xi^m K_n(\xi)\,d\xi = 2^{m-1}[(n-1)!\,p^{m-n+1}/(m-n+1) + \text{odd powers of } p] + O(p^{m+n+1}\log p). \quad (49\text{a})$$

In the latter case a logarithmic term occurs inside the square bracket, due to the term in ξ^{-1} in the expansion of $\xi^m K_n(\xi)$, and so

$$-\int_p^\infty \xi^m K_n(\xi)\,d\xi = 2^{n-1}[(n-1)!\,p^{m-n+1}/(m-n+1) + \text{even powers of } p + O(\log p) + O(p^{m+n+1}\log p). \quad (49\text{b})$$

In both cases

$$p^{m+2}K_n'(p) = -2^{n-1}[n(n-1)!\,p^{m-n+1} + \text{a power series in } p] + O(p^{m+n+1}\log p) \quad (50)$$

Dividing equations (49) by equation (50) and interpreting, we obtain the asymptotic expressions

$$\left.\begin{aligned} I_{m,n}(x) &\sim \frac{1}{n(n-m-1)}[1+O(x^{-2n})], \quad \text{if } m+n \text{ is even;} \\ &\sim \frac{1}{n(n-m-1)}[1+O(x^{m-n+1})], \quad \text{if } m+n \text{ is odd.} \end{aligned}\right\} \quad (51)$$

When $n \leqslant m+1$ the same technique can be used but the results must be obtained individually. The asymptotic expressions for $I_{0,n}(x)$ and $I_{1,n}(x)$ are summarized below;

$$\left.\begin{aligned} I_{0,0}(x) &\sim \tfrac{1}{2}\pi x - \log 2x - 1 + \tfrac{1}{4}\pi x^{-1} + x^{-2}(\log 2x - 1/6) - \tfrac{1}{2}\pi x^{-3}(\log 2x - 17/16), \\ I_{0,1}(x) &\sim \log 2x + x^{-2}(\log 2x - 3/2) + (15/4x^4)(\log 2x - 5/4), \\ I_{0,n}(x) &\sim 1/n(n-1), \quad (n > 1); \end{aligned}\right\} \quad (52)$$

$$\left.\begin{aligned} I_{1,0}(x) &= \tfrac{1}{2}x^2, \quad \text{exactly},\\ I_{1,1}(x) &\sim \tfrac{1}{2}\pi x - 1 - \tfrac{1}{4}\pi x^{-1} - \tfrac{2}{3}x^{-2} + \tfrac{1}{2}\pi x^{-3}(\log 2x - 19/8),\\ I_{1,2}(x) &\sim \tfrac{1}{2}\log 2x + \tfrac{1}{4} + O(x^{-4}),\\ I_{1,n}(x) &\sim 1/n(n-2), \quad (n > 2). \end{aligned}\right\} \tag{53}$$

Although these expressions are strictly true asymptotically as $x \to \infty$, it is found that when $n = 1$ they are unexpectedly inaccurate for moderately large values of x, due to certain exponential terms which occur in the expressions for $W_n(x)$ when $n > 0$ (cf. Randall 1955). For example, it may easily be shown that

$$\mathbf{I}_{0,1}(p) = p^{-2} - p^{-3}\mathbf{W}_1(p),$$

and $W_1(x)$ contains the term

$$-2\mathscr{R}\{\alpha_1 e^{\alpha_1 x}/(1+\alpha_1^2)\},$$

where $\alpha_1 = -0{\cdot}6453 + 0{\cdot}5012i$. We should therefore add to the asymptotic expression given in (52) for $I_{0,1}(x)$ the additional term

$$-e^{-0{\cdot}645x}(0{\cdot}571\cos 0{\cdot}501x + 2{\cdot}174\sin 0{\cdot}501x),$$

and this is found to improve the accuracy considerably. Thus, for $x = 3$ the unmodified formula gives $I_{0,1} = 1{\cdot}85$; the additional term gives 1·53, the correct value being 1·58. The corresponding values when $x = 4$ are 2·13, 2·00 and 2·02 respectively.

In the same way it is found that the correction

$$e^{-0{\cdot}645x}(1{\cdot}797\cos 0{\cdot}501x - 1{\cdot}756\sin 0{\cdot}501x)$$

should be added to the expression for $I_{1,1}(x)$ in (53); again this gives an accuracy of about 1 % when $x = 4$. The corresponding corrections for $n > 1$ are much smaller and may usually be neglected.

4. The effect of body distortions on overall wing forces and moments

The most immediate application of the results of the preceding section is to the problem of calculating the effect of small body distortions on the overall lift and pitching and rolling moments of the wings of combinations for which the body is approximately cylindrical in the neighbourhood of its junction with the wings.

Consider a thin wing mounted centrally at zero incidence in the plane $z = 0$ on a quasi-cylindrical body of the type considered previously, such that the intersection of the wing leading edges and the body is in the plane $x = 0$; the leading edges must be supersonic, the trailing edges straight and parallel to Oy, and the aspect ratio must be sufficiently large that the Mach lines $X = \beta(Y - R_0)$ do not intersect the wing tips (see figure 1). We suppose for simplicity that the body is exactly cylindrical and circular both for $x < 0$ and for $z < 0$, but that the upper half of the body is distorted for $x > 0$ in such a way that the slope of a meridian section at the point (X, θ) is $\eta(X, \theta)$. Then only the cosine terms of the Fourier expansion (7) need be taken; thus

$$\eta(x, \theta) = \sum_{n=0}^{\infty} A_n(x)\cos n\theta, \quad (x > 0,\ z > 0), \tag{54}$$

where
$$A_0(x) = \pi^{-1}\int_0^\pi \eta(x,\theta)d\theta,$$
and
$$A_n(x) = 2\pi^{-1}\int_0^\pi \eta(x,\theta)\cos n\theta\, d\theta, \quad (n>0).$$

The disturbance potential for $z > 0$ is then given in operational form by equation (9) with the sine terms omitted, since the downwash in the plane of the wing is then zero ; for $z < 0$, ϕ is identically zero forward of the influence region of the trailing edge since, the leading edges are supersonic. The pressure on the upper surface of the wing is similarly given by equation (10), with $\theta = 0$ $(y > 0)$ or $\theta = \pi (y < 0)$.

Referring now to § 3.1, we deduce at once that the *lift* on the starboard wing is given by
$$L = \int_0^C (\partial L/\partial X)\, dX = \beta R_0 \int_0^c (\partial L/\partial X)\, dx,$$
where C is the root chord and $c = C/\beta R_0$. Thus the operational form for the lift is
$$L(c)/(R_0^2 q_0) \supset -2 \sum_{n=0}^{\infty} \mathbf{I}_{0,n}(p)\mathbf{A}_n(p) \tag{55}$$
from equation (17), in which x has to be replaced by c, so that p now corresponds to d/dc.

Interpreting equation (55) by means of the product theorem, we get
$$L(c)/(R_0^2 q_0) = -2 \sum_{n=0}^{\infty} \int_{\xi=0-}^{c} I_{0,n}(c-\xi)\, dA_n(\xi). \tag{56}$$

The rolling moment M_x on this wing is similarly given by
$$M_x/(R_0^3 q_0) = -2 \sum_{n=0}^{\infty} \int_{\xi=0-}^{c} I_{1,n}(c-\xi)\, dA_n(\xi). \tag{57}$$

The pitching moment M_y about Oy is given by
$$M_y = -\int_0^C X(\partial L/\partial X)\, dX = -\beta^2 R_0^2 \int_0^c x(\partial L/\partial X)\, dx. \tag{58}$$

Now, from equation (17),
$$\frac{1}{R_0 q_0}\frac{\partial L}{\partial X} = -\frac{2}{\beta}\sum_{n=0}^{\infty}\frac{d}{dx}\int_{\xi=0-}^{x} I_{0,n}(x-\xi)\, dA_n(\xi).$$

Substituting in equation (58) and integrating by parts, we get
$$M_y/(\beta R_0^3 q_0) = 2\sum_{n=0}^{\infty}\Big[c\int_{\xi=0-}^{c} I_{0,n}(c-\xi)\, dA_n(\xi) - \\ -\int_0^c dx \int_{\xi=0}^{x} I_{0,n}(x-\xi)\, dA_n(\xi)\Big].$$

Reversing the order of integration in the second term, we have
$$M_y/(\beta R_0^3 q_0) = 2\sum_{n=0}^{\infty}\int_{\xi=0-}^{c}[cI_{0,n}(c-\xi) - I_{0,n}^{(-1)}(c-\xi)]\, dA_n(\xi), \tag{59}$$
where $I_{0,n}^{(-1)}(\xi) = \int_0^\xi I_{0,n}(\eta)\, d\eta$.

The total lift and moment on *both* wings produced by a body distortion of the specified type can now be obtained by carrying out the same process for the port wing (taking $\theta = \pi$ in equation (10)), and adding to the corresponding result for the starboard wing. Thus

$$L/(R_0^2 q_0) = -4 \sum_{n=0}^{\infty} \int_{\xi=0-}^{c} I_{0,2n}(c-\xi)\, dA_{2n}(\xi), \tag{60}$$

$$M_x/(R_0^3 q_0) = -4 \sum_{n=0}^{\infty} \int_{\xi=0-}^{c} I_{1,2n+1}(c-\xi)\, dA_{2n+1}(\xi), \tag{61}$$

and

$$M_y/(\beta R_0^3 q_0) = 4 \sum_{n=0}^{\infty} \int_{\xi=0-}^{c} [cI_{0,2n}(c-\xi) - I_{0,2n}^{(-1)}(c-\xi)]\, dA_{2n}(\xi). \tag{62}$$

5. The effect of wing-body interference on overall wing forces

We shall now consider the application of the theory of § 3 to certain problems in wing–body interference. In the present paper we shall treat only the simplest case, that of a rectangular wing mounted symmetrically on an infinite circular cylinder whose axis is in the direction of the free stream. The basic theory is due to Nielsen (1955), and only a brief summary will be given here.

5.1. *Nielsen's theory of wing–body interference*

The total disturbance potential ϕ of the combination may be expressed in the form

$$\phi = \phi_W + \phi_I, \tag{63}$$

where ϕ_W is the potential of the wing alone (which is usually taken to be continued in a suitable manner through the body), and ϕ_I is the additional potential necessary to satisfy the boundary conditions on the body; in the present case there is of course no 'body alone' potential.

Provided that the wing leading edge is supersonic, then ahead of the influence region of the trailing edge the flows over the upper and lower surfaces of the wing are independent, and the interference potential may be expanded as a Fourier series similar to equation (4):

$$\varphi_I(p, r, \theta) = \sum_{n=0}^{\infty} a_n(p) K_n(pr) \cos n\theta. \tag{64}$$

Here the coefficients $a_n(p)$ may differ according as z is greater or less than zero, and the coefficients of odd order will vanish when the wing is symmetrical about the plane $y = 0$, as is normally the case. The potential ϕ, defined by equation (63), then automatically satisfies the boundary condition in the plane of the wing since $\partial\phi_I/\partial\theta = 0$ there. In order that the boundary condition of zero normal velocity at the body surface may be satisfied, it is necessary that

$$\partial\phi_I/\partial r = -\partial\phi_W/\partial r = -R_0 u_{rW}, \tag{65}$$

when $r = 1$. Here u_{rW} is the radial component of velocity on the surface of the body due to the wing alone. Let u_{rW} be expanded as the Fourier series

$$u_{rW} = -U \sum_{n=0}^{\infty} A_{2n}(x) \cos 2n\theta, \tag{66}$$

where

$$A_0 = -\pi^{-1} \int_0^{\pi} u_{rW} U^{-1} d\theta,$$

and

$$A_{2n} = -2\pi^{-1} \int_0^{\pi} u_{rW} U^{-1} \cos 2n\theta \, d\theta, \quad (n > 0).$$

Then, if $\mathbf{A}_{2n}(p)$ are the operational forms of $A_{2n}(x)$, the boundary condition (65) gives

$$a_{2n}(p) = r_0 U \mathbf{A}_{2n}(p) / \{p K'_{2n}(p)\}, \tag{67}$$

so that the interference pressure field in the plane of the wing is given by

$$C_{pI} = (P_I - P_0)/q_0 = -2\beta^{-1} \sum_{n=0}^{\infty} \mathbf{A}_{2n}(p) K_{2n}(pr)/K'_{2n}(p). \tag{68}$$

It is clear that the ratio $-u_{rW}/U$ corresponds to the body slope η considered in § 2, § 3 & § 4, so that the effect of the interference between the flow field of the wing and the body is equivalent (so far as the wing is concerned) to a simple distortion of the body of the type considered previously. The remainder of the theory is thus exactly as given in § 2 and need not be discussed further here; some results concerning the detailed pressure distribution are given by Nielsen (1955).

5.2. *Lift and pitching moment of rectangular wings on a cylindrical body*

Consider a thin rectangular wing, whose aspect ratio is sufficiently large that the Mach lines from the leading edge of the wing–body junction do not intersect the wing tips (the aspect ratio A of the *exposed* wing must exceed $2/\beta$). The wing is mounted at incidence α on a cylinder of radius r_0 which is at zero incidence, so that the wing leading edge coincides with the y-axis. The flow field of the wing alone is then two-dimensional in the appropriate region of the body, and consists simply of a constant downwash U over the region $|z| < z$. We have therefore $u_{rW} = -U\alpha \sin\theta$ for $|z| < x$, while $u_{rW} = 0$ elsewhere; and the Fourier coefficients $A_{2n}(x)$ are given, for $0 \leqslant \theta \leqslant \pi$, by

$$A_0 = 2\alpha\pi^{-1} \int_0^{\phi} \sin\theta \, d\theta$$

and

$$A_{2n} = 4\alpha\pi^{-1} \int_0^{\phi} \sin\theta \cos 2n\theta \, d\theta, \quad (n \geqslant 1)$$

where $\phi = \sin^{-1} x$ if $x \leqslant 1$, and $\phi = \frac{1}{2}\pi$ if $x > 1$. Thus (see Nielsen 1955)

$$A_{2n}(x) = \alpha f_{2n}(x),$$

where

$$\left.\begin{aligned} f_0(x) &= (2/\pi)\{1-(1-x^2)^{1/2}\} && \text{if } x \leqslant 1, \\ &= 2/\pi && \text{if } x \geqslant 1; \\ \text{and, for } n \geqslant 1, & \\ f_{2n}(x) &= \frac{2}{\pi}\left[\frac{\cos(2n-1)\phi}{2n-1} - \frac{\cos(2n+1)\phi}{2n+1} - \frac{2}{4n^2-1}\right] && \text{if } x \leqslant 1, \\ &= -4/\pi(4n^2-1) && \text{if } x > 1. \end{aligned}\right\} \quad (69)$$

5.2.1. *Lift*

The total lift increment due to interference, which is twice that on the upper surface of the wing, is given by equation (60) as

$$L_I(c)/(R_0^2 q_0) = -8 \sum_{n=0}^{\infty} \int_{\xi=0-}^{c} I_{0,2n}(c-\xi)\, dA_{2n}(\xi). \quad (70)$$

Since $A_{2n}(x)$ are all continuous functions of x such that $A_{2n}(0) = 0$, equation (70) gives

$$L_I/(R_0^2 q_0 \alpha) = -8 \sum_{n=0}^{\infty} \int_0^c I_{0,2n}(c-\xi) f'_{2n}(\xi)\, d\xi, \quad (71\text{a})$$

where

$$\left.\begin{aligned} f'_0(x) &= 2\pi^{-1} x(1-x^2)^{-1/2} \\ f'_{2n}(x) &= 4\pi^{-1} x(1-x^2)^{-1/2} \cos 2n\phi \quad (n \geqslant 1) \end{aligned}\right\}$$

when $0 \leqslant x \leqslant 1$, and $f'_{2n}(x) = 0$ for $x \geqslant 1$. Thus, if $c \geqslant 1$,

$$L_I/(R_0^2 q_0 \alpha) = -8 \sum_{n=0}^{\infty} \int_0^1 I_{0,2n}(c-\xi) f'_{2n}(\xi)\, d\xi. \quad (71\text{b})$$

We may write this result

$$L_I/(R_0^2 q_0 \alpha) = -8G(c),$$

where

and

$$\left.\begin{aligned} G(c) &= \sum_{n=0}^{\infty} G_{2n}(c), \\ G_{2n}(c) &= \int_0^c I_{0,2n}(c-\xi) f'_{2n}(\xi)\, d\xi. \end{aligned}\right\} \quad (72)$$

It will be noticed that the functions $f'_{2n}(x)$ become infinite like $(1-x)^{-1/2}$ when $x = 1$; because of this singularity the numerical integrations up to $\xi = 1$ were completed by using a formula of the type given by Jeffreys & Jeffreys (1956, § 9.092):

$$\int_0^{4h} (\alpha x^{-1/2} + \beta x^{1/2} + \gamma x^{3/2} + \delta x^{5/2})\, dx = h(5\cdot 994 y_1 - 8\cdot 836 y_2 + 8\cdot 974 y_3 - 1\cdot 854 y_4), \quad (73)$$

where y_n is the value of the integrand at $x = nh$.

The functions G_0, G_2, G_4 and G_6 have been computed for a range of c from 0 to 5, and are given in table 2.

c	G_0	G_2	G_4	G_6
0	0	0	0	0
0·2	0	0·0001	0·0001	0·0001
0·4	0·0006	0·0012	0·0009	0·0005
0·6	0·0032	0·0052	0·0023	−0·0003
0·8	0·0102	0·0135	0·0008	−0·0049
1·0	0·0258	0·0234	−0·0098	−0·0065
1·2	0·0583	0·0214	−0·0164	−0·0027
1·4	0·1088	0·0015	−0·0165	0·0010
1·6	0·1743	−0·0294	−0·0133	0·0015
1·8	0·2528	−0·0663	−0·0087	0
2·0	0·3424	−0·1046	−0·0052	−0·0020
2·2	0·4417	−0·1407	−0·0036	−0·0026
2·4	0·5494	−0·1723	−0·0039	−0·0017
2·6	0·6645	−0·1979	−0·0053	−0·0008
2·8	0·7861	−0·2170	−0·0068	−0·0008
3·0	0·9135	−0·2298	−0·0079	−0·0010
3·2	1·0461	−0·2371	−0·0082	
3·4	1·1833	−0·2397	−0·0081	
3·6	1·3245	−0·2388	−0·0076	
3·8	1·4695	−0·2355	−0·0071	
4·0	1·6179	−0·2309	−0·0068	
4·2	1·7694	−0·2258		
4·4	1·9234	−0·2208		
4·6	2·0801	−0·2163		
4·8	2·2390	−0·2127		
5·0	2·4000	−0·2100		
∞		−0·2122	−0·0071	0·0012

Table 2. The functions $G_{2n}(c)$.

When c is large, an asymptotic expansion for $G(c)$ may easily be obtained from the results given in equation (52) for $I_{0,n}(x)$. We have, when $c > 1$, integrating equation (72) by parts,

$$G_0(c) = (2/\pi)\left\{I_{0,0}(c) - \int_0^1 I'_{0,0}(c-\xi)(1-\xi^2)^{-1/2}\,d\xi\right\},$$

and since $I_{0,0}(x) \sim \tfrac{1}{2}\pi x - \log 2x - 1 + \tfrac{1}{4}\pi/x + O(\log x/x^2)$, it follows that

$$G_0(c) \sim c - (2/\pi)\log 2c - (2/\pi + \tfrac{1}{4}\pi) + c^{-1} + O(c^{-2}\log c). \tag{74}$$

Similarly, we find that

$$G_{2n}(c) \sim -2/n\pi(4n^2-1)(2n-1) \tag{75}$$

when $n \geqslant 1$; and since it can be shown that

$$\sum_{n=1}^{\infty} [n(4n^2-1)(2n-1)]^{-1} = \tfrac{1}{8}\pi^2 + \tfrac{1}{2} - 2\log 2,$$

it follows that

$$G(c) \sim c - (2/\pi)\log \tfrac{1}{2}c - (3/\pi + \tfrac{1}{2}\pi) + c^{-1} + O(c^{-2}\log c). \tag{76}$$

It is found that for $c = 5$ the asymptotic formula (76) gives a result which differs by less than 5% from that obtained by direct computation, so that it provides a satisfactory method of extending the calculations to any desired value of c. Greater difficulty is experienced for small values of c, due to slow convergence of the series $\sum G_{2n}'(c)$. A first approximation as c approaches zero can be found, following Nielsen (1955) by considering the body as a vertical boundary on which is a given distribution of sources corresponding to the normal velocity produced by the wing. In this way we find that

$$G(c) \doteqdot c^3/9\pi \qquad \text{when } c \to 0. \tag{77}$$

The use of four terms only of the series $\sum G_{2n}(c)$ in determining the function $G(c)$ appears to be adequate, with a probable error not exceeding about 2%,

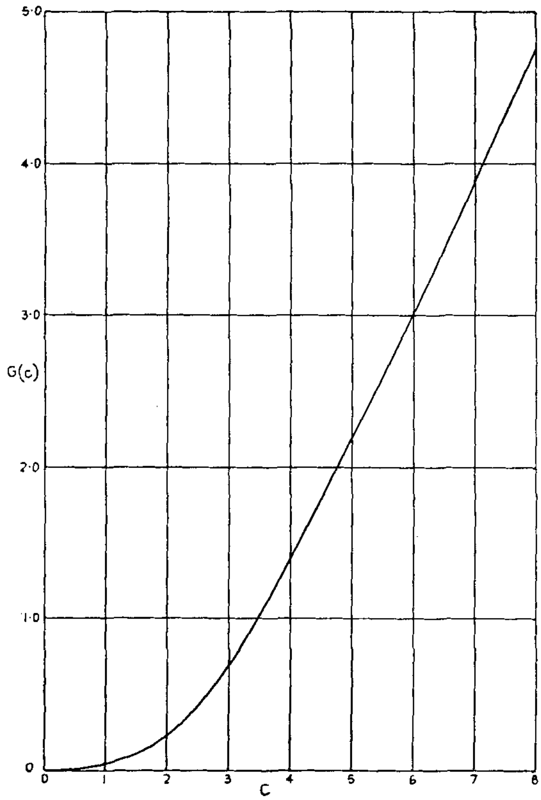


Figure 3. The function $G(c)$.

for $c > 2$; for smaller values of c the accuracy decreases rapidly, while the formula (96) cannot be expected to apply for $c > \frac{1}{2}$. There is thus an intermediate range in which $G(c)$ has to be estimated graphically, but this is not expected to introduce any serious error. The function $G(c)$ is shown in figure 3.

Nielsen (1955) expresses his results for the effect of interference on the total lift of rectangular wings of finite aspect ratio in the form of a coefficient k_{LW} defined as the ratio of the lift on the exposed wings in combination with the body to that on the exposed wings joined together in the absence of the body. It can easily be shown that, provided the exposed aspect ratio A exceeds $2/\beta$, so that tip effects do not enter into the interference calculations, the ratio k_{LW} may be expressed in terms of the function $G(c)$ as

$$k_{LW} = 1 - 4G(c)/c^2(2\beta A - 1). \tag{78}$$

The quantity $4G(c)/3c^2$, which gives the value of $(1-k_{LW})$ when $\beta A = 2$, is shown in figure 4 and compared with the corresponding result obtained by Nielsen (1955). The agreement between the two methods is reasonably good for the smaller values of c (less than about 4), but Nielsen has overestimated the magnitude of the interference effect when c is large. This

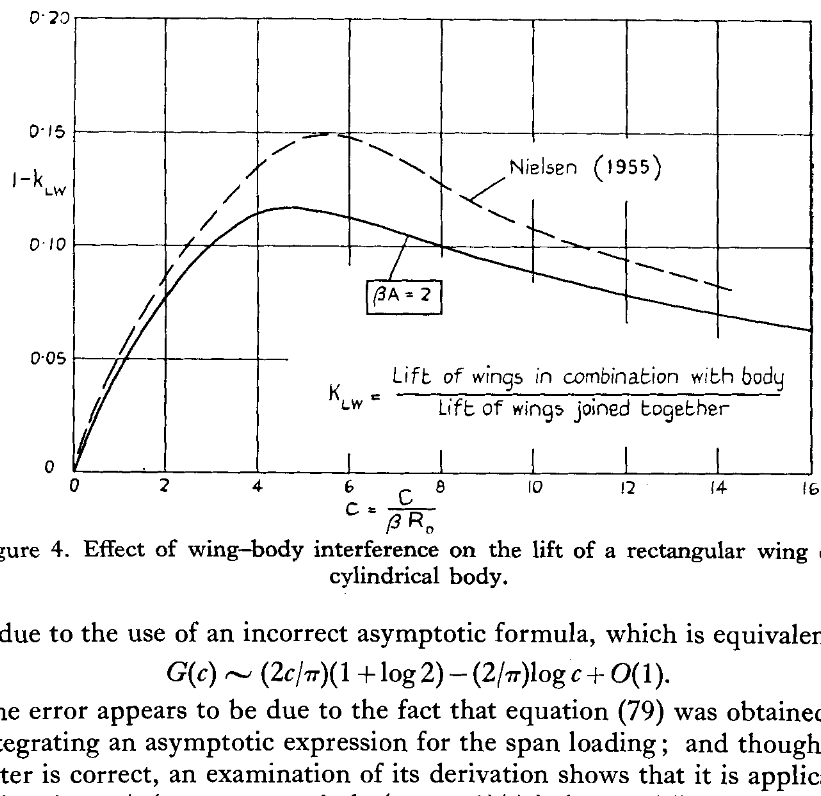


Figure 4. Effect of wing–body interference on the lift of a rectangular wing on a cylindrical body.

is due to the use of an incorrect asymptotic formula, which is equivalent to

$$G(c) \sim (2c/\pi)(1+\log 2) - (2/\pi)\log c + O(1). \tag{79}$$

The error appears to be due to the fact that equation (79) was obtained by integrating an asymptotic expression for the span loading; and though the latter is correct, an examination of its derivation shows that it is applicable only when x/r (or more precisely $(x-r+1)/r$) is large. Thus, even when c is large, this expression is only correct in a region near the body, and it is not permissible to integrate it in a spanwise direction to obtain the total lift.

5.2.2. *Pitching moment and centre of pressure*

The increase M_{yI} in nose-up pitching moment about the leading edge due to interference is found by means of equation (62) to be

$$M_{yI}/(\beta R_0^3 q_0 \alpha) = 8 \sum_{n=0}^{\infty} \int_0^c [cI_{0,2n}(c-\xi) - I_{0,2n}^{(-1)}(c-\xi)] f_{2n}'(\xi)\, d\xi, \tag{80}$$

and this is equivalent to

$$M_{yI}/(\beta R_0^3 q_0 \alpha) = 8[cG(c) - G^{(-1)}(c)], \tag{81}$$

where

$$G^{(-1)}(c) = \int_0^c G(x)\, dx.$$

Using equation (72) for the lift decrement, combined with the standard linearized theory for wings of finite span, it can easily be shown that the position of the centre of pressure is given in terms of the chord by

$$\frac{X_{C.P.}}{C} - \frac{1}{2} = -\left[\frac{1 + 12c^{-2}G(c) - 24c^{-3}G^{(-1)}(c)}{6k_{LW}(2\beta A - 1)}\right] \tag{82}$$

where k_{LW} is given by (78). The asymptotic expression when c is large is

$$\frac{X_{C.P.}}{C} - \frac{1}{2} \sim -\left[\frac{1 + 24c^{-2}(\log \frac{1}{2}c + \frac{1}{4}\pi^2 - \frac{1}{2})}{6k_{LW}(2\beta A - 1)}\right]. \tag{83}$$

The variation of the position of the centre of pressure with c is shown in figure 5 for $\beta A = 2$; again Nielsen overestimates the forward movement for c greater than about 4.

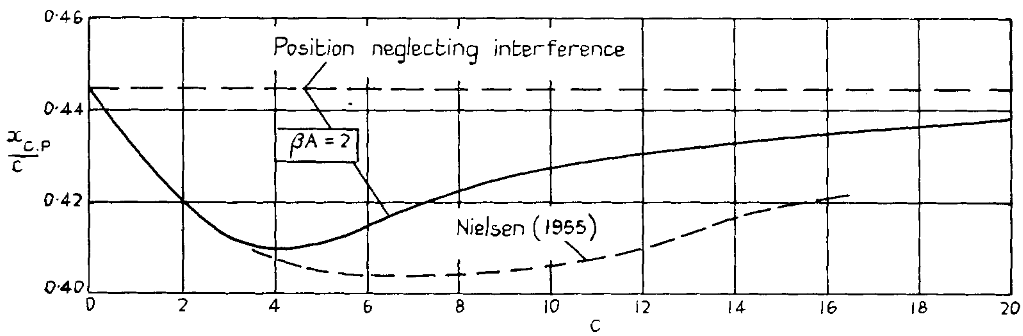


Figure 5. Effect of wing–body interference on position of centre of pressure for a rectangular wing on a cylindrical body.

5.3. *The drag at zero lift of rectangular wings on a cylindrical body*

The theory of § 5.1 can easily be extended to give the drag of any rectangular wing of symmetrical cross-section at zero incidence. We suppose that the shape of the wing section is given by the slope

$$dZ/dX = \delta(X/C) \qquad \text{for } z > 0. \tag{84}$$

The wing alone produces, for $z > 0$, an upwash field

$$u_{zW} = U\delta(X/C - \beta Z/C).$$

(No connection is intended with the Dirac delta function; $\delta(x)$ may be an arbitrary function of x in $0 \leqslant x \leqslant 1$ and may have a finite number of discontinuities in that range.) This upwash field has a component normal to the body surface:

$$\left.\begin{aligned} u_{rW} &= U|\sin\theta|\delta(x/c - |\sin\theta|/c) \quad \text{when } -\phi \leqslant \theta \leqslant \phi \\ &\qquad\qquad \text{and } \pi - \phi \leqslant \theta \leqslant \pi + \phi, \\ &= 0 \text{ elsewhere,} \end{aligned}\right\} \tag{85}$$

where as before $\phi = \sin^{-1} x$ if $0 \leqslant x \leqslant 1$, and $\phi = \frac{1}{2}\pi$ if $x > 1$.

The Fourier components of $-u_{rW}/U$ are thus

$$\left.\begin{aligned} A_0 &= -\frac{2}{\pi}\int_0^{\phi} \sin\theta\,\delta\left(\frac{x-\sin\theta}{c}\right) d\theta, \\ \text{and}\qquad A_{2n} &= -\frac{4}{\pi}\int_0^{\phi} \cos 2n\theta \sin\theta\,\delta\left(\frac{x-\sin\theta}{c}\right) d\theta. \end{aligned}\right\} \tag{86}$$

If we write $\sin\theta = \xi$ and $\cos 2n\theta = C_{2n}(\xi)$ (a polynomial of degree $2n$ in ξ), then

$$A_{2n}(x) = -\frac{4}{\pi}\int_0^x \frac{\xi}{(1-\xi^2)^{1/2}} C_{2n}(\xi)\delta\left(\frac{x-\xi}{c}\right) d\xi.$$

But (cf. equation (69))

$$\begin{aligned} f_{2n}(x) &= \frac{4}{\pi}\int_0^{\phi} \sin\theta\cos 2n\theta\, d\theta, \\ &= \frac{4}{\pi}\int_0^x \xi\frac{C_{2n}(\xi)}{(1-\xi^2)^{1/2}} d\xi. \end{aligned}$$

Hence

$$\begin{aligned} A_{2n}(x) &= -\int_0^x f'_{2n}(\xi)\delta\left(\frac{x-\xi}{c}\right) d\xi \\ &= -f_{2n}(x)\delta(0) - \frac{1}{c}\int_0^x f_{2n}(x-\xi)\delta'(\xi/c)\, d\xi. \end{aligned} \tag{87 a}$$

This may be written in Stieltjes form

$$A_{2n}(x) = -\int_{\xi=0-}^{x} f_{2n}(x-\xi)\, d\delta(\xi/c), \tag{87 b}$$

where $\delta(\xi)$ is defined to be zero for $\xi < 0$.

The total interference drag D_I is given by

$$\begin{aligned} D_I &= 4\int_{X=0}^{C} \delta(X/C)\, dX \int_{R=R_0}^{\infty} (P_I - P_0)\, dR \\ &= 2R_0^2\rho_0 U^2 \int_{x=0}^{c} \delta(x/c)\, dx \int_{r=1}^{\infty} \beta C_{pI}(x,r)\, dr. \end{aligned} \tag{88}$$

Now we consider the operational form

$$\int_1^{\infty} \beta C_{pI}\, dr = 2\sum_{n=0}^{\infty} p\mathbf{A}_{2n}(p) I_{0,2n}(p) \tag{89}$$

$$= 2\frac{d}{dx}\sum_{n=0}^{\infty}\int_0^x I_{0,2n}(x-\xi)A'_{2n}(\xi)\, d\xi, \tag{90}$$

interpreting equation (89) by the product theorem and making use of the fact that $A_{2n}(0) = 0$. Substituting in equation (88) and integrating by parts, we get

$$\begin{aligned} D_I/(R_0^2 q_0) &= 8\int_{x=0}^{c} \delta(x/c)\, dx \frac{d}{dx}\int_0^x \sum_{n=0}^{\infty} I_{0,2n}(x-\xi)A'_{2n}(\xi)\, d\xi \\ &= -8\int_{x=0}^{c+} d\delta(x/c) \int_0^x \sum_{n=0}^{\infty} I_{0,2n}(x-\xi)A'_{2n}(\xi)\, d\xi, \end{aligned} \tag{91}$$

where again $\delta(\xi)$ is defined to be zero for $\xi > 1$.

But, from (87),

$$\int_0^x I_{0,2n}(x-\xi)A'_{2n}(\xi)\,d\xi = -\int_{\xi=0}^{x} I_{0,2n}(x-\xi)\,d\xi \int_{\zeta=0-}^{\xi} f'_{2n}(\xi-\zeta)\,d\delta(\zeta/c)$$

$$= -\int_{\zeta=0-}^{x} d\delta(\zeta/c) \int_{\xi=\zeta}^{x} I_{0,2n}(x-\xi)f'_{2n}(\xi-\zeta)\,d\xi$$

$$= -\int_{\xi=0-}^{x} G_{2n}(x-\xi)\,d\delta(\xi/c), \tag{92}$$

where $G_{2n}(x)$ is defined above (equation (72)). Substituting in equation (91), we have finally

$$D_I/(R_0^2 q_0) = 8\int_{x=0}^{c+} d\delta(x/c) \int_{\xi=0-}^{x} G(x-\xi)\,d\delta(\xi/c) \tag{93 a}$$

It is necessary to use the Stieltjes form (93 a) if $\delta(\xi)$ has any discontinuities in the range $0 < \xi < 1$, but if $\delta(\xi)$ is continuous in this range, then the extended form of this equation is

$$D_I/(8R_0^2 q_0) = c^{-2}\int_0^c \delta'(x/c)\,dx \int_0^x \delta'(\xi/c)G(x-\xi)\,d\xi +$$

$$+c^{-1}\left\{\delta(0)\int_0^c \delta'(x/c)G(x)\,dx - \delta(1)\int_0^x \delta'(1-x/c)G(x)\,dx - \right.$$

$$\left. -\delta(0)\delta(1)G(c)\right\}. \tag{93 b}$$

For a section symmetrical about $x/c = \frac{1}{2}$, this simplifies to

$$D_I/(8R_0^2 q_0) = c^{-2}\int_0^c \delta'(x/c)\,dx \int_0^x \delta'(\xi/c)G(x-\xi)\,d\xi +$$

$$+2c^{-1}\delta(0)\int_0^c \delta'(x/c)G(x)\,dx + \delta^2(0)G(c). \tag{93 c}$$

For a wing of biconvex parabolic section and thickness ratio τ, we have

$$\delta(x/c) = 2\tau(1-2x/c), \tag{94}$$

and equation (93 c) gives

$$D_I/(R_0^2 q_0) = 8\tau^2\left\{16c^{-2}\int_0^c dx \int_0^x G(\xi)\,d\xi - 16c^{-1}\int_0^c G(x)\,dx + 4G(c)\right\}$$

$$= 8\tau^2\left\{16c^{-2}\int_0^c (c-x)G(x)\,dx - 16c^{-1}\int_0^c G(x)\,dx + 4G(c)\right\}$$

$$= -32c^{-2}\tau^2\left\{4\int_0^c x\,G(x)\,dx - c^2 G(c)\right\}. \tag{95}$$

The interference drag ratio k_{DW}, defined in a similar way to k_{LW} by

$$k_{DW} = \frac{D_W + D_I}{D_W} = \frac{\textit{Total wave drag}}{\textit{Wave drag of exposed wing alone}},$$

is thus given by

$$k_{DW} = 1 - \frac{6}{\beta A c^2}\left\{\frac{4}{c^2}\int_0^c xG(x)\,dx - G(c)\right\} \tag{96}$$

for a wing of biconvex section.

The corresponding result for a symmetrical double wedge section (for which the Stieltjes form (93 a) has to be used) is

$$k_{DW} = 1 - \frac{2}{\beta A c^2} \{4G(\tfrac{1}{2}c) - G(c)\}. \tag{97}$$

These results are shown in figure 6, where the drag ratio k_{DW} is plotted against the equivalent chord–radius ratio $c = C/(\beta r_0)$, for wings of exposed aspect ratio $2/\beta$. As c increases from zero, k_{DW} at first rises from 1·0 to a maximum of about 1·03, reached when c is between 3 and 4; it then decreases steadily, being again equal to 1·0 for c between 6 and 7, and reaches a minimum of about 0·97 when c is 15, after which it increases slowly towards the asymptotic value 1·0. The difference between the results for the two wing sections is very small.

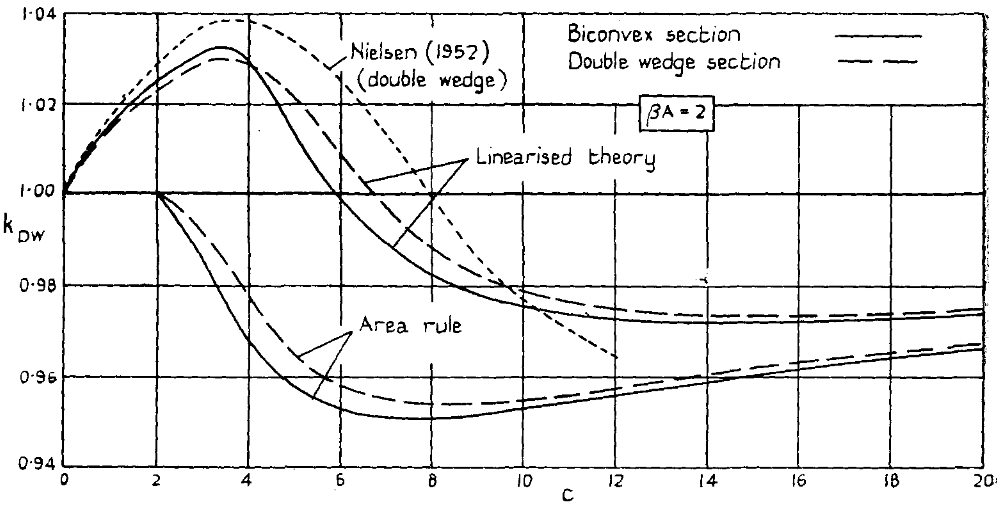


Figure 6. Effect of wing–body interference on wave drag of a rectangular wing on a cylindrical body.

The results for a double wedge section may be compared with those obtained by Nielsen (1955); there is good qualitative agreement but again Nielsen has overestimated the magnitude of the interference effect, for the same reason that has been suggested in the case of lift.

A qualitative explanation of the behaviour of the interference drag can most easily be given by considering a wing of double wedge section. The forward half of the wing produces a compression wave which is in effect reflected from the body as an expansion acting over the influence region of the junction of the wing leading edge with the body. This expansion causes a thrust on the forward half wing and a drag on the rear half. Similarly, the rear half wing produces an expansion which is reflected as a compression over the smaller influence region of the junction of the half-chord line with the body; this produces a further thrust. The balance between these components of thrust and drag depends on the value of c;

and it may be seen from consideration of the corresponding results for a *single* wedge (which are qualitatively similar to those for the lift ratio k_{LW} (figure 4)) that for small values of c the drag on the rear half outweighs the two components of thrust, while as c is increased the balance is gradually altered until for large values of c there is a resultant decrease in the total drag as compared with that of the wing alone. This balance between opposing forces of thrust and drag also explains why the magnitude of the effect of wing–body interference on drag is in this case very much less than on lift; thus the drag correction does not exceed $\pm 3\%$, while the lift correction has a maximum value of about 12% (both for $\beta A = 2$).

An interesting general result for large values of c may be obtained by substituting the asymptotic expression (95) directly into the interference drag formulae (93). It may easily be shown that the leading terms of the resultant asymptotic expression for D_I are

$$D_I/8R_0^2 q_0 \sim -c\int_0^1 [\delta(\xi)]^2\, d\xi + \pi^{-1}\log c\{\delta^2(0)+\delta^2(1)\}+O(1), \qquad (98)$$

provided that $\delta(\xi)$ is continuous in $0 < \xi < 1$.

Now the wave drag of the exposed wing panels joined together is given by

$$D_W/q_0 = 4\beta^{-1}AC^2\int_0^1 [\delta(\xi)]^2\, d\xi,$$

so that

$$D_W/(8R_0^2 q_0) = \tfrac{1}{2}\beta Ac^2\int_0^1 [\delta(\xi)]^2\, d\xi.$$

Hence

$$k_{DW} \sim -\frac{2}{\beta Ac}\left\{1-\frac{1}{\pi c}\log c\,\frac{[\delta^2(0)+\delta^2(1)]}{\int_0^1 [\delta(\xi)]^2\, d\xi}+O(c^{-1})\right\}. \qquad (99)$$

It is noteworthy that the first term in this expression is independent of the section shape. For a biconvex section, equation (99) gives

$$k_{DW} \sim -2/\beta Ac\{1-(6/\pi c)\log c+O(c^{-1})\}, \qquad (100)$$

which may be verified from equation (96).

It may also be shown from equation (97) that the first two terms of the corresponding asymptotic expression for k_{DW} for a double wedge section are in fact identical with (100), but it appears from equation (99) that this is not true for a general section shape.

Although it has been shown here that the effect on the total wave drag at zero lift of the interference between rectangular wings of moderate aspect ratio and cylindrical bodies is small from a practical point of view, the problem considered here is of some fundamental interest because it provides an example in which it is possible to compare the results of true linear theory with those of the supersonic area rule.

6. Comparison with the supersonic area rule

It has recently been realized that the supersonic area rule due to Jones (1953) for the wave drag of wing–body combinations is not strictly correct, even within the limitations of the linearized theory, except for a very restricted class of objects, which it is not at present possible to define at all clearly. This question has been discussed in detail by Lomax & Heaslet (1956), who give several examples in which the area rule can be shown to lead to considerable errors; the problem considered in § 5 provides a further example.

The area rule is in effect based on two fundamental assumptions, which are to some extent inter-related: (*a*) that the flow round a symmetrical wing–body combination at zero lift may be represented by a distribution of simple sources over the plane of the wings and along the axis of the body; and (*b*) that the total (linearized) velocity potential for the combination is equal to the sum of the potentials for the exposed wing and the body taken separately, each in the absence of the other.

These assumptions are clearly incorrect in the case of wings mounted on a cylindrical body. For such combinations assumption (*b*) implies that the velocity potential is simply equal to that of the *exposed* wings alone; and this will not in general satisfy the condition of zero normal velocity on the body, so that an additional axial distribution of poles *and multipoles* has to be added, thus violating assumption (*a*).

In order to make a more direct comparison with Nielsen's method, the area-rule potential may be thought of as made up of the potential of the complete wing continued through the body, together with the negative of the potential due to the portion of the wing blanketed by the body; and the latter should be equal to the interference potential ϕ_I of § 5.1. Although these potentials are to some extent similar, it is obviously impossible in general for them to be completely equivalent. That this is so in the case of rectangular wings may be verified indirectly by comparison of the interference drags obtained by the two methods.

Consider a rectangular wing of overall span $2s$ and unit chord, mounted on a cylindrical body of radius R_0. The oblique area distribution $S(\xi, \theta)$ of the exposed wing, defined as the projection on planes perpendicular to Ox of the area of a section by the plane

$$X = \xi + \beta Y \cos\theta, \tag{101}$$

is clearly given by

$$S(\xi, \theta) = S(\xi, \theta, s) - S(\xi, \theta, R_0), \tag{102}$$

where $S(\xi, \theta, s)$ is the corresponding area distribution for a complete rectangular wing of span $2s$.

The derivative of $S(\xi, \theta, s)$ with respect to ξ is given by

$$S'(\xi, \theta, s) = 2\beta^{-1} \sec\theta\{f(\xi + \beta s\cos\theta) - f(\xi - \beta s\cos\theta)\}, \tag{103}$$

where $Z = f(X)$ is the equation of the wing section (cf. Lock 1957), and $f(X)$ is defined to be zero outside the wing chord.

The supersonic area rule gives for the overall wave drag

$$D/q_0 = -\pi^{-2} \int_0^{\pi/2} d\theta \iint d\xi d\eta \log|\xi - \eta| S''(\xi, \theta) S''(\eta, \theta), \tag{104}$$

where primes denote differentiation with respect to ξ or η; the double integral is taken over the range of ξ (or η) for which $S''(\xi, \theta)$ does not vanish.

Substituting for $S(\xi, \theta)$ from equation (102), we see that

$$D = D(s) + D(R_0) - 2D_i(s, R_0), \tag{105}$$

where $D(s)$ is the drag of a rectangular wing of span $2s$ with the given section, and D_i is defined by

$$q_0^{-1}D(s, R_0) = -\pi^{-2}\int_0^{\pi/2} d\theta \iint d\xi d\eta \log|\xi-\eta| S''(\xi,\theta,s)S''(\eta,\theta,R_0) \tag{106}$$

$$= -4\pi^{-2}\beta^{-2}\int_0^{\pi/2} \sec^2\theta\, d\theta \iint d\xi\, d\eta \log|\xi-\eta| \times$$

$$\times \{f'(\xi+\sigma_1) - f'(\xi-\sigma_1)\}\{f'(\eta+\sigma_2) - f'(\eta-\sigma_2)\}, \tag{107}$$

where $\sigma_1 = \beta s \cos\theta$ and $\sigma_2 = \beta R_0 \cos\theta$. Linear transformations of the variables ξ and η reduce this to the form

$$q_0^{-1}D_i(s, R_0) = -4\pi^{-2}\beta^{-2}\int_0^{\pi/2} \sec^2\theta\, d\theta \int_0^1\int_0^1 dx\, dy\, f'(x)f'(y) \times$$

$$\times \log|\{(x-y)^2 - (\sigma_1-\sigma_2)^2\}/\{(x-y)^2 - (\sigma_1+\sigma_2)^2\}|; \tag{108}$$

and by comparing this with the corresponding expression for $D(s)$ (see Lock 1957) it may easily be shown that

$$D_i = D\{\tfrac{1}{2}(s+R_0)\} - D\{\tfrac{1}{2}(s-R_0)\}. \tag{109}$$

Now if $s \geqslant \frac{1}{2}\beta^{-1}$, it is known that

$$D(s)/q_0 = 2sC_{D_0},$$

where C_{D_0} is the two-dimensional wave drag coefficient for the given wing section; and if the aspect ratio $A = 2(s-R_0)$ of the exposed wing is restricted, as in § 5.3, to be greater than $2/\beta$, then equation (109) gives simply

$$D_i/q_0 = 2R_0 C_{D_0}.$$

Substituting in equation (124), we get

$$D/q_0 = 2(s-R_0)C_{D_0} + \{q^{-1}D(R_0) - 2R_0 C_{D_0}\}. \tag{110}$$

The first term represents the drag of the exposed wing panels joined together, so that the interference drag ratio k_{DW} is obtained as

$$k_{DW} = 1 + \frac{2R_0}{2(s-R_0)}\left(\frac{C_{D_1}}{C_{D_0}} - 1\right) = 1 + \frac{2}{\beta A c}\left(\frac{C_{D_1}}{C_{D_0}} - 1\right) \tag{111}$$

where C_{D_1} is the drag coefficient of the portion of the wing blanketed by the body, whose aspect ratio is $2/(\beta c)$ in the notation of § 5. (This result has also been obtained by Sheppard (1957).) Equation (111) suggests that $k_{DW} = 1$ for $0 \leqslant c \leqslant 2$, and that $k_{DW} < 1$ for $c > 2$. This can easily be seen qualitatively by considering the physical assumption on which the area rule is based: that the flow field is equivalent to that of the exposed wing panels with the body removed. When c is less than 2, there is no interaction between the two wings and the drag is unaltered; as soon as c becomes greater than 2, the compression wave from the inner leading edge of one wing acts near the trailing edge of the other to give a reduction in total drag.

The value given by equation (111) for k_{DW} according to the area rule has been calculated for wings of biconvex and double wedge section, using the results of Harmon (1947); we obtain

$$k_{DW} = 1 - \frac{2}{\pi\beta Ac} 2\cos^{-1}\frac{2}{c} - \frac{1}{c}\left[\left(6 - \frac{4}{c^2}\right)\cosh^{-1}\tfrac{1}{2}c - \left(1 - \frac{4}{c^2}\right)^{1/2}\right] \quad (112)$$

for a biconvex section, and

$$k_{DW} = 1 - \frac{2}{\pi\beta Ac}\{f(c) - f(\tfrac{1}{2}c)\},$$

where

$$f(c) = \tfrac{1}{2}(c^2-4)^{1/2} + (2/c)\cosh^{-1}\tfrac{1}{2}c - 2\cos^{-1}(2/c) \quad (113)$$

for a symmetrical double wedge section.

These results are included in figure 6 for comparison with those of the accurate linearized theory of § 5. It is clear that for values of c less than about 6, the area rule is completely erroneous in predicting the effect of wing–body interference in the present case, and that not until c exceeds 10 is there any quantitative agreement with linearized theory. The latter point was to be expected from previous work on the subject; it has frequently been stated, and recently proved rigorously by Fraenkel (1958) that the area rule is always true for combinations of the type considered here, provided that the ratio corresponding to c of the present paper is large. In fact it may easily be verified that the first two terms of the asymptotic expansions for k_{DW} when c is large, obtained from equations (112) and (113), are identical with the corresponding terms in equations (96) and (97) (cf. equations (100) *et seq.*).

It must again be emphasized that the actual numerical difference between the results of the area rule and of exact linearized theory is not large for the example considered here; the ratio *Drag* (*Linearized Theory*)/*Drag* (*Area Rule*) has a maximum of about $1 + 0{\cdot}12/\beta A$ when $c \doteqdot 4$, for values (> 2) of βA which are covered by the present theory. But there is little doubt that for smaller aspect ratios the error in estimating the wave drag of the wings by the area rule will continue to increase and may become of practical importance.

7. Conclusions

An extension has been developed of the linearized theory of Nielsen (1955) and Randall (1955) which enables more rapid and accurate calculations than hitherto possible to be made of the overall forces and moments acting on a restricted class of wings mounted on quasi-cylindrical bodies. Two principal examples have been studied in detail.

The first of these is concerned with the changes produced by small distortions of an otherwise cylindrical body on the static stability of thin centrally mounted wings at zero incidence, subject to the condition that the trailing edge is straight and unswept and that the leading edges are supersonic.

The second application is to the problem of wing–body interference at supersonic speeds, in particular to the example previously considered by Nielsen (1955) of a rectangular wing mounted on a long cylindrical body of circular cross-section, the latter being at zero incidence; both methods are applicable only if the exposed ratio exceeds $2/\beta$. The effect of interference on lift and pitching moment is found to be in reasonably good agreement with Nielsen's results for small values of the equivalent chord–radius ratio $c = -C/\beta R_0$, but for values of c greater than about 4 the theory of the present paper, which predicts smaller interference effects than does Nielsen's (due to a slight error in the latter), should be the more accurate.

A new formula has been derived for the effect of interference on wave-drag at zero incidence which enables this to be computed rapidly and accurately for a rectangular wing of arbitrary cross-section. The results for double wedge and biconvex parabolic sections are compared with those of the supersonic area rule, and it is found that there is poor agreement until the ratio c is greater than about 10; and this is a value that is seldom attained in practice. The numerical discrepancy in the *total* wave drag of the wings is not large for aspect ratios covered by the present method, but may well become more serious for values of βA much less than 2; although, since only the drag of the wings has been considered here, the area rule may still give reasonable accuracy for a typical complete wing–body combination when the drag of the fore- and after-bodies is taken into account.

It is clear that further comparisons of the type given above are desirable before a true assessment of the accuracy of the supersonic area rule can be made. One possible extension of the methods of the present paper would be to study the effects of body modifications on the wave drag of combinations with rectangular wings, and it may also be possible to design body shapes to give an overall reduction in drag; these shapes and the corresponding drag values could then be compared with the predictions of the area rule.

The work described above was carried out in the Aerodynamics Division of the National Physical Laboratory and is published by permission of the Director of the Laboratory.

References

FRAENKEL, L. E. 1958 The wave drag of wing-quasi-cylinder combinations at zero incidence. *Aero. Quart.* **9**, 55.

HARMON, S. M. 1947 Theoretical supersonic wave drag of untapered sweptback and rectangular wings at zero lift, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 1449.

JEFFREYS, H. & JEFFREYS, B. S. 1956 *Methods of Mathematical Physics*, 3rd Ed. Cambridge University Press.

JONES, R. T. 1953 Theory of wing–body drag at supersonic speeds, *Nat. Adv. Comm. Aero., Wash., Rep.* no. 1284.

LOCK, R. C. 1957 A note on the application of the supersonic area rule to the determination of the wave drag of rectangular wings, *J. Fluid Mech.* **2**, 575.

LOMAX, H. & HEASLET, M. A. 1956 Recent developments in the theory of wing–body wave drag, *J. Aero. Sci.* **23**, 1061.

MERSMAN, W. A. 1954 Numerical calculation of certain inverse Laplace transforms *Proc. Intern. Cong. Math.*, Amsterdam, **2.**

NIELSEN, J. N. & PITTS, W. C. 1952 Wing–body interference at supersonic speeds with an application to combinations with rectangular wings, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 2677.

NIELSEN, J. N. 1955 Quasi-cylinder theory of wing–body interference at supersonic speeds and comparison with experiment, *Nat. Adv. Comm. Aero., Wash., Rep.* no. 1252.

NIELSEN, J. N. 1957 Tables of characteristic functions for solving boundary value problems of the wave equation with application to supersonic interference, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no 3873.

POL, B. VAN DER & BREMMER, H. 1950 *Operational Calculus*. Cambridge University Press.

RANDALL, D. G. 1955 Supersonic flow past quasi-cylindrical bodies of almost circular cross-section, *Roy. Air. Est. Tech. Note Aero.* no. 2404. (*Aero. Res. Council., Lond., Tech. Note* no. 18492.)

SHEPPARD, L. M. 1957 The wave drag of exposed rectangular wings, *Roy. Air. Est. Tech. Note Aero.* no. 2494. (*Aero. Res. Counc., Lond., Tech. Note.* no. 19318.)

WARD, G. N. 1955 *Linearised Theory of Steady High-speed Flow.* Cambridge University Press.

WATSON, G. N. 1944 *A Treatise on the Theory of Bessel Functions*, 2nd Ed. Cambridge University Press.